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Some Results on Novikov–Poisson Algebras

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Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. A Novikov–Poisson algebra is a Novikov algebra with a compatible commutative associative algebraic structure, which was introduced to construct the tensor product of two Novikov algebras. In this paper, we commence a study of finite-dimensional Novikov–Poisson algebras. We show the commutative associative operation in a Novikov–Poisson algebra is a compatible global deformation of the associated Novikov algebra. We also discuss how to classify Novikov–Poisson algebras. And as an example, we give the classification of 2-dimensional Novikov–Poisson algebras.

KEY WORDS: Novikov algebra; Novikov–Poisson algebra; deformation.

1. INTRODUCTION

Poisson brackets of hydrodynamic type were introduced and studied in Refs. (Balinskii and Novikov, 1985; Dubrovin and Novikov, 1983, 1984; Zel'manov, 1987):

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

In Ref. (Balinskii and Novikov, 1985), in order to define a local translationally invariant Lie algebra arising from Eq. (1.1), a new algebra A with a bilinear product $(x, y) \to xy$ was introduced to satisfy the following equations:

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.2}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2, \tag{1.3}$$

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for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.4}$$

The algebra A satisfying Eqs. (1.2)–(1.3) is called a "Novikov algebra" by Osborn (Osborn, 1992a,b, 1994; Xu, 1996, 1997). It also has a close connection to some Hamiltonian operators in the formal variational calculus (Gel'fand and Diki, 1975, 1976; Gel'fand and Dorfman, 1979; Xu, 1995, 2000) and some nonlinear partial differential equations, such as KdV equations (Dubrovin and Novikov, 1983; Gel'fand and Diki, 1975, 1976). On the other hand, Novikov algebras are a special class of left-symmetric algebras which only satisfy Eq. (1.3). Left-symmetric algebras are nonassociative algebras arising from the study of affine manifolds, affine structures, and convex homogeneous cones (Bai and Meng, 2000; Burde, 1998; Kim, 1986; Vinberg, 1963).

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y]^- = xy - yx, \tag{1.5}$$

defines a (subadjacent) Lie algebra A^-. Let L_x, R_x denote the left and right multiplication, respectively, i.e., $L_x(y) = xy$, $R_x(y) = yx$, $\forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. A Novikov algebra A is called right-nilpotent or transitive if every R_x is nilpotent. The transitivity corresponds to the completeness of the affine manifolds in geometry (Kim, 1986; Vinberg, 1963).

In general, it is a natural way to construct new examples of algebras by means of the tensor product. However, as in the case of Lie algebras, usually the tensor product of two arbitrary Novikov algebras is not a Novikov algebra any more. Following the ideas of Lie–Poisson algebras, Xu in Ref. (Xu, 1996) introduced the concept of Novikov–Poisson algebras. He proved that there exists a Hamiltonian superoperator associated to a Novikov–Poisson algebra $(A, \cdot, *)$ with an identity element 1 in $(A, \cdot)$ such that $1 * 1 = 2$ (Xu, 2000). Moreover, the tensor product of two Novikov–Poisson algebras is still a Novikov–Poisson algebra (Xu, 1996). In Ref. (Xu, 1997) X. Xu gave the classification of finite-dimensional Novikov–Poisson algebras whose Novikov algebras are simple over a field with characteristic $p > 2$ and the infinite dimensional Novikov–Poisson algebras whose Novikov algebras are simple with an idempotent element over a field with characteristic 0. However, there is not a general theory of Novikov–Poisson algebras until now, even in low dimensions. On the other hand, the classification of Novikov algebras in low dimensions is only up to dimension 3 (Bai and Meng, 2001a). Therefore, it is quite useful to obtain some interesting Novikov algebras in higher dimensions by the tensor products of Novikov algebras.

In this paper, we will commence a study of finite-dimensional Novikov–Poisson algebras. The paper is organized as follows. In Section 2, we briefly discuss some basic properties and an important construction of Novikov–Poisson algebras. In Section 3, we discuss the relation between the deformation theory of Novikov algebras and Novikov–Poisson algebras. In Section 4, we discuss how to classify Novikov–Poisson algebras. As an example, we give a classification of Novikov–Poisson algebras in dimension 2. We also briefly discuss the 4-dimensional Novikov algebras which are the tensor product of two Novikov algebras in dimension 2. In Section 5, we give some conclusion based on the discussion in the previous sections.

2. NOVIKOV–POISSON ALGEBRAS

A Novikov–Poisson algebra A is a vector space with two operations "$\cdot$, $*$" such that $(A, \cdot)$ forms a commutative associative algebra (which may not have an identity element) and $(A, *)$ forms a Novikov algebra (which satisfies Eqs. (1.2)–(1.3)) with the compatible identities:

$$(x \cdot y) * z = x \cdot (y * z) = y \cdot (x * z), \tag{2.1}$$

$$(x * y) \cdot z - x * (y \cdot z) = (y * z) \cdot z - y * (x \cdot z), \tag{2.2}$$

for $x, y, z \in A$. If $(A, \cdot, *)$ is a Novikov–Poisson algebra, we also say $(A, *)$ is a Novikov algebra over the commutative associative algebra $(A, \cdot)$ or $(A, \cdot)$ is a commutative associative algebra over the Novikov algebra $(A, *)$.

From Ref. (Xu, 1996), for two Novikov–Poisson algebras $(A_1, \cdot, *)$ and $(A_2, \cdot, *)$, we can define two operations $\cdot$ and $*$ on $A_1 \otimes A_2$ such that $(A_1 \otimes A_2, \cdot, *)$ forms a Novikov–Poisson algebra by

$$(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot y_1) \otimes (x_2 \cdot y_2) \tag{2.3}$$

$$(x_1 \otimes x_2) * (y_1 \otimes y_2) = (x_1 * y_1) \otimes (x_2 \cdot y_2) + (x_1 \cdot y_1) \otimes (x_2 * y_2), \tag{2.4}$$

for $x_i, y_i \in A_i$, $i = 1, 2$.

In fact, Eqs. (2.3)–(2.4) can be expressed in terms of linear transformations (matrices), which will be useful in application. Let L^*_x, R^*_x denote the left and right multiplication of the Novikov algebra $(A, *)$, respectively and $L^{\cdot}_x$, $R^{\cdot}_x$ denote the left and right multiplication of the commutative associative algebra $(A, \cdot)$, respectively, that is

$$L^*_x(y) = x * y,\ R^*_x(y) = y * x,\ L^{\cdot}_x(y) = x \cdot y,\ R^{\cdot}_x(y) = y \cdot x,\ \forall x, y \in A. \tag{2.5}$$

Then Eqs. (2.3)–(2.4) are equivalent to the following equations:

$$L^*_{a\otimes b} = L^*_a \otimes L^{\cdot}_b + L^{\cdot}_a \otimes L^*_b;\ R^*_{a\otimes b} = R^*_a \otimes R^{\cdot}_b + R^{\cdot}_a \otimes R^*_b; \tag{2.6}$$

$$L^{\cdot}_{a\otimes b} = L^{\cdot}_a \otimes L^{\cdot}_b;\ R^{\cdot}_{a\otimes b} = R^{\cdot}_a \otimes R^{\cdot}_b, \tag{2.7}$$

for any $a \in A_1$, $b \in A_2$ and the $\otimes$ appearing in the right hand of the above equations is the tensor product of two linear transformations over the tensor spaces. Recall that the tensor product of two linear transformations f_1 and f_2 is given by $(f_1 \otimes f_2)$

$(a \otimes b) = f_1(a) \otimes f_2(b)$. In fact, for any $a, c \in A_1$, $b, d \in A_2$, we have

$$L^*_{a\otimes b}(c \otimes d) = (a \otimes b) * (c \otimes d) = (a * c) \otimes (b \cdot d) + (a \cdot c) \otimes (b * d)$$
$$= L^*_a(c) \otimes L^{\cdot}_b(d) + L^{\cdot}_a(c) \otimes L^*_b(d) = (L^*_a \otimes L^{\cdot}_b + L^{\cdot}_a \otimes L^*_b)(c \otimes d).$$

Similarly, the other equations also can be obtained from Eqs. (2.3)–(2.4). On the other hand, it is also easy to obtain Eqs. (2.3)–(2.4) from Eqs. (2.6)–(2.7).

Example 2.1. Let $\{e^l_1, e^l_2\}(l = 1, 2)$ be a basis of Novikov–Poisson algebras $(A_l, *, \cdot)$ respectively (the classification is given in Section 4) and $e^l_i * e^l_j = \sum_{k=1}^n c^{lk}_{ij} e_k$ and $e^l_i \cdot e^l_j = \sum_{k=1}^n d^{lk}_{ij} e_k$. Then by Eq. (2.6), we can obtain a 4-dimensional Novikov algebra defined by (under the basis $\{e^1_1 \otimes e^2_1, e^1_1 \otimes e^2_2, e^1_2 \otimes e^2_1, e^1_2 \otimes e^2_2\}$)

$$R^*_{e^1_i \otimes e^2_j} = \begin{pmatrix} c^{11}_{1i} & c^{12}_{1i} \\ c^{11}_{2i} & c^{12}_{2i} \end{pmatrix} \otimes \begin{pmatrix} d^{21}_{1j} & d^{22}_{1j} \\ d^{21}_{2j} & d^{22}_{2j} \end{pmatrix} + \begin{pmatrix} d^{11}_{1i} & d^{12}_{1i} \\ d^{11}_{2i} & d^{12}_{2i} \end{pmatrix} \otimes \begin{pmatrix} c^{21}_{1j} & c^{22}_{1j} \\ c^{21}_{2j} & c^{22}_{2j} \end{pmatrix}$$
$$= \begin{pmatrix} c^{11}_{1i}d^{21}_{1j} + d^{11}_{1i}c^{21}_{1j} & c^{11}_{1i}d^{22}_{1j} + d^{11}_{1i}c^{22}_{1j} & c^{12}_{1i}d^{21}_{1j} + d^{12}_{1i}c^{21}_{1j} & c^{12}_{1i}d^{22}_{1j} + d^{12}_{1i}c^{22}_{1j} \\ c^{11}_{1i}d^{21}_{2j} + d^{11}_{1i}c^{21}_{2j} & c^{11}_{1i}d^{22}_{2j} + d^{11}_{1i}c^{22}_{2j} & c^{12}_{1i}d^{21}_{2j} + d^{12}_{1i}c^{21}_{2j} & c^{12}_{1i}d^{22}_{2j} + d^{12}_{1i}c^{22}_{2j} \\ c^{11}_{2i}d^{21}_{1j} + d^{11}_{2i}c^{21}_{1j} & c^{11}_{2i}d^{22}_{1j} + d^{11}_{2i}c^{22}_{1j} & c^{12}_{2i}d^{21}_{1j} + d^{12}_{2i}c^{21}_{1j} & c^{12}_{2i}d^{22}_{1j} + d^{12}_{2i}c^{22}_{1j} \\ c^{11}_{2i}d^{21}_{2j} + d^{11}_{2i}c^{21}_{2j} & c^{11}_{2i}d^{22}_{2j} + d^{11}_{2i}c^{22}_{2j} & c^{12}_{2i}d^{21}_{2j} + d^{12}_{2i}c^{21}_{2j} & c^{12}_{2i}d^{22}_{2j} + d^{12}_{2i}c^{22}_{2j} \end{pmatrix}$$

Let $(A_1, *, \cdot)$ and $(A_2, *, \cdot)$ be two Novikov–Poisson algebras. Then by Eqs. (2.3)–(2.4) or Eqs. (2.6)–(2.7), it is easy to obtain the following conclusions:

(a) $(A_1 \otimes A_2, *, \cdot)$ is isomorphic (see Section 4) to $(A_2 \otimes A_1, *, \cdot)$ through the exchange operator $\tau : A_1 \otimes A_2 \to A_2 \otimes A_1$, $\tau(a \otimes b) = b \otimes a$, $\forall a \in A_1, b \in A_2$.

(b) If both $(A_1, *)$ and $(A_2, *)$ are commutative Novikov algebras, then $(A_1 \otimes A_2, *)$ given by Eq. (2.4) is commutative.

(c) Let $(A_1, *)$ and $(A_2, *)$ be two associative Novikov algebras. Then $(A_1 \otimes A_2), *)$ given by Eq. (2.4) is associative if for any x, y, z in A_i, we have

$$x * (y \cdot z) = (x * y) \cdot z. \tag{2.8}$$

(d) Recall that a Lie–Poisson algebra $(A, [,], \cdot)$ is a vector space with two operations $[,], \cdot$ such that $(A, [,])$ forms a Lie algebra and $(A, \cdot)$ forms a commutative associative algebra with the compatible condition:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \forall x, y, z \in A. \tag{2.9}$$

Then for a Novikov–Poisson algebra $(A, *, \cdot)$, $(A, [,]^-, \cdot)$ is a Lie–Poisson algebra if and only if

$$x * (y \cdot z) = x \cdot (y * x), \forall x, y, z \in A \tag{2.10}$$

where $[x, y]^- = x * y - y * x$ is given by Eq. (1.5).

In general, it is not easy to construct the Novikov–Poisson algebras. However, there is a class of Novikov–Poisson algebras obtained from commutative associative algebras (Xu, 1997): Let $(A, \cdot)$ be a commutative associative algebra and D be its derivation. Then with the following product:

$$a * b = a \cdot Db, \tag{2.11}$$

$(A, *)$ becomes a Novikov algebra. This Novikov algebraic structure was firstly given by S. Gel'fand and $(A, *)$ is proved to be transitive in Ref. (Bai and Meng, 2001b). Moreover, Xu proved that $(A, *, \cdot)$ is a Novikov–Poisson algebra.

3. NOVIKOV–POISSON ALGEBRAS AND THE DEFORMATION OF NOVIKOV ALGEBRAS

In Ref. (Bai and Meng, 2001b), we give a deformation theory of Novikov algebras, which has a close relation with Novikov–Poisson algebras. At first, for self-contained, we briefly introduce the deformation theory of Novikov algebras given in Ref. (Bai and Meng, 2001b). Let $(A, *)$ be a Novikov algebra, and $g_p : A \times A \to A$ be a bilinear product defined by

$$g_q(a, b) = a * b + qG_1(a, b) + q^2 G_2(a, b) + q^3 G_3(a, b) + \dots \tag{3.1}$$

where G_i are bilinear products with $G_0(a, b) = a * b$. We call (A_q, g_q) a deformation of $(A, *)$ if (A_q, g_q) is a family of Novikov algebras. In particular, we call G_1 an global deformation if the deformation is given by

$$g_q(a, b) = a * b + qG_1(a, b), \tag{3.2}$$

that is, $G_2 = G_3 = \dots = 0$. G_1 is a global deformation if and only if

$$G_1(a, b * c) - G_1(a * b, c) + G_1(b * a, c) - G_1(b, a * c) + a * G_1(b, c)$$
$$- G_1(a, b) * c + G_1(b, a) * c - b * G_1(a, c) = 0; \tag{3.3}$$

$$G_1(a, b) * c - G_1(a, c) * b + G_1(a * b, c) - G_1(a * c, b) = 0. \tag{3.4}$$

$$G_1(a, b * c) - G_1(a * b, c) + G_1(b * a, c) - G_1(b, a * c) + a * G_1(b, c)$$
$$- G_1(a, b) * c + G_1(b, a) * c - b * G_1(a, c) = 0; \tag{3.5}$$

$$G_1(a, b) * c - G_1(a, c) * b + G_1(a * b, c) - G_1(a * c, b) = 0. \tag{3.6}$$

Moreover, G_1 is in the space of 2-cocycles, and G_1 is called a compatible global deformation if G_1 is commutative. Any Novikov algebra and its compatible global deformation have the same subadjacent Lie algebra. A global deformation is called special if the family of Novikov algebras (A_q, g_q) defined by Eq. (3.2) is mutually isomorphic for $q \neq 0$. We proved that the Novikov algebras in dimension ≤ 3 can be realized as the algebras defined through Eq. (2.11) and their compatible global deformations in Refs. (Bai and Meng, 2001b,c). Moreover, *Claim*. Let $(A, *, \cdot)$ be a Novikov-Poisson algebra. Then both $G_1(a, b) = a \cdot b$ and $G_1(a, b) = x \cdot a \cdot b$ for a fixed $x \in A$ are the compatible global deformations of $(A, *)$. Hence, $(A, *_x)$ becomes a Novikov algebra by the following product:

$$a *_x b = a * b + x \cdot a \cdot b. \tag{3.7}$$

where $x \in \mathbf{F}$ or $x \in A$.

In fact, for $G_1(a, b) = a \cdot b$, Eq. (3.3) holds since

$$a \cdot (b * c) - (a * b) \cdot c + (b * a) \cdot c - b \cdot (a * c) = b * (a \cdot c) - a * (b \cdot c);$$

$$a * (b \cdot c) - (a \cdot b) * c + (b \cdot a) * c - b * (a \cdot c) = a * (b \cdot c) - b * (a \cdot c),$$

and Eq. (3.4) holds since

$$(a \cdot b) * c - (a \cdot c) * b + (a * b) \cdot c - (a * c) \cdot b = 0.$$

Obviously Eq. (3.5) and Eq. (3.6) hold due to commutativity. Similarly, Eq. (3.3)–Eq. (3.6) hold for $G_1(a, b) = x \cdot a \cdot b$ for a fixed $x \in A$. □

Actually, Eq. (3.7) with a fixed $x \in A$ is firstly given by Xu (Xu, 1997). Moreover, for a Novikov–Poisson algebra $(A, *, \cdot)$, there is a class of Novikov–Poisson algebras $(A, *_x, \cdot_y)$ with the following products (Xu, 1997):

$$a *_x b = a * b + x \cdot a \cdot b, a \cdot_y b = y \cdot a \cdot b, \tag{3.8}$$

for any fixed $x, y \in A$ or $\mathbf{F}$. Moreover, combining Eq. (3.8) with Eq. (2.11), we have the following corollary:

Corollary. (Xu, 1997) *Let $(A, \cdot)$ be a commutative associative algebra and D be its derivation. Then for any fixed elements $x, y \in A$ or $\mathbf{F}$, $(A, *_x, \cdot_y)$ becomes a Novikov–Poisson algebra with the following products:*

$$a *_x b = a \cdot Db + x \cdot a \cdot b, a \cdot_y b = y \cdot a \cdot b. \tag{3.9}$$

In fact, the Novikov algebras obtained through by Eq. (3.9) for $x \in \mathbf{F}$ is given by Filipov (Filipov, 1989). On the other hand, there exist Novikov–Poisson algebras which cannot be obtained from Eq. (3.9), which can be seen from the next section.

4. ON THE CLASSIFICATION OF NOVIKOV–POISSON ALGEBRAS

Since there are two operations in a Novikov–Poisson algebra, it is not easy to obtain the classification of Novikov–Poisson algebras in the sense of isomorphism. Two Novikov–Poisson algebras $(A_i, *, \cdot)$ are isomorphic if and only if there exists a linear isomorphism $f : A_1 \to A_2$, such that

$$f(a * b) - f(a) * f(b),\ f(a \cdot b) = f(a) \cdot f(b), \forall a, b \in A_1. \tag{4.1}$$

Thus we often need to discuss two algebraic isomorphisms simultaneously. Moreover, it is obvious that a linear transformation of a Novikov–Poisson algebra is an (algebraic) isomorphism (called an automorphism) if and only if it is an automorphism of both the Novikov algebra and the commutative algebra, that is, it is in the intersection of the automorphism groups of the Novikov algebra and the commutative algebra.

Usually, we fix an algebra system which has been classified at first and then we classify another algebraic structure which is compatible with the former. In general, we need the following three steps:

Step 1: Classify one algebra system with structure constants;
Step 2: For the fixed algebra system, find the compatible structure constants of the second algebra system;
Step 3: Classify those compatible structure constants of the second algebra system. Here, we would like to point out that the corresponding linear transformations describing the isomorphic relations between different structure constants of the second algebra system must be in the automorphism group of the first algebra system.

For a Novikov–Poisson algebra, because the structure of the commutative associative algebra is much simpler than that of the Novikov algebra, we can give the classification of Novikov–Poisson algebras as the classification of the compatible commutative associative algebras for the fixed Novikov algebras whose classification in low dimensions has been given in Ref. (Bai and Meng, 2001a).

Let $\{e_i\}$ be a basis of a Novikov–Poisson algebra $(A, *, \cdot)$. Then $(A, *, \cdot)$ is determined by the (form) characteristic matrix given as

$$(\mathcal{A}, *) = \begin{pmatrix} \sum_{k=1}^{n} c_{11}^{k} e_k & \dots & \sum_{k=1}^{n} c_{1n}^{k} e_k \\ \dots & \dots & \dots \\ \sum_{k=1}^{n} c_{n1}^{k} e_k & \dots & \sum_{k=1}^{n} c_{nn}^{k} e_k \end{pmatrix},$$

$$(\mathcal{A}, \cdot) = \begin{pmatrix} \sum_{k=1}^{n} d_{11}^{k} e_k & \dots & \sum_{k=1}^{n} d_{1n}^{k} e_k \\ \dots & \dots & \dots \\ \sum_{k=1}^{n} d_{n1}^{k} e_k & \dots & \sum_{k=1}^{n} d_{nn}^{k} e_k \end{pmatrix}, \tag{4.2}$$

where $e_i * e_j = \sum_{k=1}^{n} c_{ij}^{k} e_k$ and $e_i * e_j = \sum_{k=1}^{n} d_{ij}^{k} e_k$.

For a fixed $(\mathcal{A}, *)$, the elements in $(\mathcal{A}, \cdot)$ should satisfy the following equations:

$$d_{ij}^{p} = d_{ji}^{p}, \sum_{l=1}^{n} d_{ij}^{l} d_{lk}^{p} = \sum_{l=1}^{n} d_{jk}^{l} d_{il}^{p},\ p = 1, \dots, n; \tag{4.3}$$

$$\sum_{l=1}^{n} d_{ij}^{l} c_{lk}^{p} = \sum_{l=1}^{n} c_{jk}^{l} d_{il}^{p},\ p = 1, \dots, n; \tag{4.4}$$

$$\sum_{l=1}^{n} \left(c_{ij}^{l} d_{lk}^{p} - d_{jk}^{l} c_{il}^{p}\right) = \sum_{l=1}^{n} \left(c_{ji}^{l} d_{lk}^{p} - d_{ik}^{l} c_{jl}^{p}\right),\ p = 1, \dots, n. \tag{4.5}$$

From the classification of Novikov algebras in dimension 2 given in Ref. (Bai and Meng, 2001a), it is easy to get the corresponding automorphism groups. Based on these results and through Eqs. (4.3)–(4.5) and direct computation, we can give the classification of Novikov–Poisson algebras in dimension 2 in the following table: ($m, n \in \mathbf{C}$)

Characteristic matrix $(\mathcal{A}, *)$	Automorphism group aut $(\mathcal{A}, *)$	Compatible characteristic matrix $(\mathcal{A}, \cdot)$	Characteristic matrix $(\mathcal{A}, \cdot)$
(T1) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $a_{11}a_{22} - a_{12}a_{22} \neq 0$	$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}, \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}, \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$
(T2) $\begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & \sqrt{a_{11}} \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 0 & me_2 \\ me_2 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & me_1 \end{pmatrix}, \begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$
(T3) $\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$
(N1) $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}, (m \geq n)$
(N2) $\begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} ne_1 & 0 \\ 0 & me_2 \end{pmatrix}$	$\begin{pmatrix} e_1 & 0 \\ 0 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & me_2 \end{pmatrix}$
(N3) $\begin{pmatrix} 0 & e_1 \\ e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 0 & me_2 \\ me_2 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$
(N4) $\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} a_{11} & 0 \\ a_{21} & 1 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}$
(N5) $\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ a_{21} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & me_1 \end{pmatrix}$
(N6) $\begin{pmatrix} 0 & e_1 \\ le_1 & e_2 \end{pmatrix}$ $l \neq 0, 1$	$\begin{pmatrix} a_{11} & 0 \\ 0 & 1 \end{pmatrix}$ $a_{11} \neq 0$	$\begin{pmatrix} 0 & me_1 \\ me_1 & ne_1 + me_2 \end{pmatrix}$	$\begin{pmatrix} 0 & me_1 \\ me_1 & me_2 \end{pmatrix}, \begin{pmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{pmatrix}$

Using a similar way, we can obtain the classification of Novikov–Poisson algebras in dimension 3.

Remark 4.1. Comparing the characteristic matrix $(\mathcal{A}, \cdot)$ in the above table with the realization of Novikov algebras in dimension 2 given in Refs. (Bai and Meng, 2001b,c), we can see that many Novikov–Poisson algebras cannot be obtained from Eq. (3.9). In fact, the Novikov–Poisson algebra which $(\mathcal{A}, *)$ is (T3) and $(A, \cdot) = \left(\begin{smallmatrix} 0 & me_1 \\ me_1 & e_1 + me_2 \end{smallmatrix}\right)$ is just an example.

Remark 4.2. We can easily obtain the characteristic matrix of a 4-dimensional Novikov algebra which is the tensor of two Novikov–Poisson algebras in dimension 2 through the above table and the formula given in Example 2.1. For example, the characteristic matrix of the tensor of type (T3) which $(\mathcal{A}, \cdot) = \left(\begin{smallmatrix} 0 & me_1 \\ me_1 & me_2 \end{smallmatrix}\right)$ and type (N5) which $(\mathcal{A}, \cdot) = \left(\begin{smallmatrix} 0 & m'e_1 \\ m'e_1 & m'e_2 \end{smallmatrix}\right)$ is

$$\begin{pmatrix} 0 & 0 & 0 & me_1 \\ 0 & 0 & 0 & me_1 + me_2 \\ 0 & (m - m')e_1 & 0 & me_3 \\ -m'e_1 & me_1 + (m - m')e_2 & 0 & m(e_3 + e_4) \end{pmatrix}$$

However, it is not easy to classify these characteristic matrices (even may be unnecessary) because for any isomorphism $F : A_1 \otimes A_2 \to A_1 \otimes A_2$ of the tensor of two Novikov–Poisson algebras $(A_i, *, \cdot)$, there may not necessarily exist two isomorphisms $f_i : A_i \to A_i$ such that $F = f_1 \otimes f_2$.

5. DISCUSSION AND CONCLUSION

From the discussion in the previous sections, we have the following conclusion:

(a) There exist nontrivial commutative associative algebras over any Novikov algebra in dimension 2.

(b) We can see that there are the same commutative associative algebras over many (nonisomorphic) Novikov algebras. One of the reasons is perhaps due to their close relations with the realization theory given in Refs. (Bai and Meng, 2001b,c).

(c) It is easy to see that the tensor of two Novikov–Poisson algebras whose Novikov algebras are transitive may not be transitive. It is an open question when the tensor product of two Novikov–Poisson algebras is still transitive.

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